ASYMPTOTICS OF A RENEWAL-LIKE RECURSION AND AN INTEGRAL EQUATION

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ABSTRACT. We consider a renewal-like recursion and prove that the solution is polynomially decaying under suitable conditions. We prove similar results for the corresponding integral equation. In both cases coefficients and functions are of more general form than in the classic cases.

1. INTRODUCTION

In this paper we examine the asympotics of a renewal-like recursion and a similar integral equation. The motivation comes from probability theory; more precisely, in a random model of publication activity [1] the asymptotic distribution of the weights of the authors satisfy such equations.

The recursion is of the form

$$(1) \quad x_n = \sum_{j=1}^{n-1} w_{n,j} x_{n-j} + r_n, \quad w_{n,j} = a_j + \frac{b_j}{n} + c_{n,j} \quad (n = 1, 2, \ldots) \, ,$$

where $w_{n,j} \geq 0$, and a, b, c are decaying at least exponentially fast. The precise assumptions are formulated later. Our goal is to prove that x_n is polynomially decaying as $n \to \infty$ under suitable conditions, and to determine the exponent.

Similar recursions are widely examined, see e.g. Milne-Thompson [11], Cooper–Frieze [4]. In those cases either the coefficients are special, or only the last m terms appear on the right-hand side for some fixed m. Now all previous terms are present, and the weights depend both on n and j.

On the other hand, omitting (b_n) and (c_n) and supposing that (a_n) is a probability distribution, we get the well-known renewal formula [6, Chapter XIII]. The asymptotics were examined in a more subtle way in [5], for instance, by weaker Tauberian type assumptions. In our case the coefficients are of more general form; however, we have stronger conditions on them. An example will show that these assumptions can not be totally omitted (see Remark 4).

The continuous counterpart is the following integral equation, which is a Volterra equation of the second kind.

$$(2) \qquad\qquad g(t) = \int_0^t w_{t,s}\, g(t-s)\, ds + r(t)$$

for $t > 0$ and $g(0) = 1$. The kernel $w_{t,s}$ is supposed to be written in the following form.

$$0 \le w_{t,s} = a(s) + \frac{b(s)}{t+d} + c_{t,s},$$

where a is a probability density function, and again, a, b, c are decreasing fast. We will show that $g(t)$ is between two polynomially decaying functions under suitable conditions, and give the exponent. In addition, assuming that g is decreasing, we will prove that $g(t)$ is polynomially decaying as $t \to \infty$. We use Laplace transforms and Tauberian theorems in this part.

Omitting b and c we get a classic renewal equation [7, Chapter XI].

In Section 2 we formulate the main results for both cases. Sections 3 and 4 contain the proofs for the discrete and the continuous cases, respectively. Section 5 contains the Laplace transform methods.

2. Main results

2.1. **The discrete recursion.** Consider the following recursion:

$$(3) \quad x_n = \sum_{j=1}^{n-1} w_{n,j} x_{n-j} + r_n, \quad w_{n,j} = a_j + \frac{b_j}{n} + c_{n,j}, \quad (n = 1, 2, \dots),$$

where $w_{n,j} \ge 0$, and a_n, b_n, $c_{n,j}$, r_n satisfy the following conditions.

(r1) $a_n \ge 0$ for $n \ge 1$, and the greatest common divisor of the set $\{n : a_n > 0\}$ is 1;

(r2) $r_n \ge 0$, and there exists such an n that $r_n > 0$;

(r3) there exists $z > 0$ such that

$$1 < \sum_{n=1}^{\infty} a_n z^n < \infty, \qquad \sum_{n=1}^{\infty} |b_n| z^n < \infty,$$

$$\sum_{n=1}^{\infty} \sum_{j=1}^{n-1} |c_{n,j}| z^j < \infty, \qquad \sum_{n=1}^{\infty} r_n z^n < \infty.$$

It is clear that $x_n \geq 0$ for $n \geq 1$.

Our theorem gives the asymptotics of (x_n). It is polynomially decaying; the exponent is also given.

Theorem 1. *Suppose that the sequence (x_n) satisfies recursion* (3), *conditions* (r1)–(r3) *hold, and (x_n) has infinitely many positive terms. Then $x_n n^{-\gamma} q^n \to C$ as $n \to \infty$, where C is a positive constant, q is the positive solution of equation $\sum_{n=1}^{\infty} a_n q^n = 1$, and*

$$\gamma = \frac{\sum_{n=1}^{\infty} b_n q^n}{\sum_{n=1}^{\infty} n a_n q^n}.$$

Remark 1. The condition on $w_{n,j}$ in recursion (3) can be modified in the following way.

$$w_{n,j} = a_j + \frac{b_j}{n-j} + c_{n,j}, \quad n = 1, 2, \ldots .$$

The difference may be added to the remainder term $c_{n,j}$, because we have

$$\sum_{n=1}^{\infty} \sum_{j=1}^{n-1} \left| \frac{b_j}{n-j} - \frac{b_j}{n} \right| y^j$$

$$= \sum_{j=1}^{\infty} |b_j| y^j \sum_{n=j+1}^{\infty} \left(\frac{1}{n-j} - \frac{1}{n} \right) = \sum_{j=1}^{\infty} |b_j| y^j \sum_{n=1}^{j} \frac{1}{n},$$

which is finite for $0 < y < z$. Since the generating function of the sequence (a_n) is left continuous at point z, there exists $y < z$ such that $\sum_{n=1}^{\infty} a_n y^n > 1$.

Remark 2. The condition that the sequence has infinitely many positive terms is necessary as the following example shows. Let $r_1 = 1$, $r_n = 0$ if $n > 1$, and $w_{n,j} = a_n \left(1 - \frac{1}{n-j} \right)$. Then we get that $x_1 = 1$, $x_2 = x_3 = \cdots = 0$.

2.2. **The integral equation.** Now we examine an integral equation, which is similar to recursion (3). Namely, let $g : \mathbb{R} \to \mathbb{R}$ be the solution of the following integral equation; we will explain later why the solution exists.

$$(4) \qquad g(t) = \int_0^t w_{t,s}\, g(t-s)\, ds + r(t)$$

for $t > 0$ and $g(0) = 1$. Here

$$0 \le w_{t,s} = a(s) + \frac{b(s)}{t+d} + c_{t,s},$$

and the following conditions hold.

- (i1) $a \in L^1[0,\infty)$ is a probability density function concentrated on the set of positive real numbers. That is, a is nonnegative almost everywhere, and $\int_0^\infty a(s)\, ds = 1$.
- (i2) $b \in L^1[0,\infty)$, and d is a positive constant.
- (i3) $r \in L^1[0,\infty)$ is a nonnegative, continuous function.
- (i4) $c : [0,\infty)^2 \to \mathbb{R}$ is (jointly) measurable, $c_{t,s}$ is integrable on $[0,t]$ with respect to s for all $t > 0$, and $\lim_{t\to\infty} c_{t,s} = 0$ for a.e. $s > 0$.
- (i5) There exists $z > 1$ such that

$$\int_0^\infty a(t) z^t\, dt < \infty, \quad \int_0^\infty |b(t)|\, z^t\, dt < \infty, \quad \text{and}$$

- (i6) $z^t \int_0^t |c_{t,s}|\, ds$ and $r(t) z^t$ are directly Riemann integrable with respect to t on $[0,\infty)$.

Recall that a nonnegative function h is directly Riemann integrable on $[0,\infty)$ (see p. 361 of [7]), if and only if it is (Riemann) integrable on every finite interval, and for all $\tau > 0$ we have

$$\sum_{n=1}^\infty \sup_{n\tau \le \theta \le (n+1)\tau} h(\theta) < \infty,$$

that is, the upper Riemann sum of h with span τ is finite. As usual, we say that a real function h is directly Riemann integrable if both its positive and negative parts are directly Riemann integrable. This is equivalent to the direct Riemann integrability of $|h|$.

Equation (4) is a nonlinear Volterra-type integral equation of the second kind. It is easy to check that all conditions of Theorem 3.2. of [10] hold for this equation in a finite interval $0 \le t \le T$. Thus, applying the theorem we get that the equation has a unique and continuous solution for all positive T. Hence $g(t)$ is defined on the set of nonnegative real numbers, and it is continuous. Since the proof of Theorem 3.2. of [10] relies on Picard approximation, and $g(0)$, w, r are all nonnegative, it is clear that $g(t)$ is nonnegative for all $t \ge 0$.

Our main results are about the asymptotics of $g(t)$ as $t \to \infty$. First we give the order of g by proving lower and upper bounds. Then assuming that g is decreasing, we will find the asymptotics of g using Laplace transforms.

Theorem 2. *Let g be the solution of equation (4). Suppose that w is nonnegative, all conditions (i1)–(i6) hold, and for all $T > 0$ there exists $t > T$ such that $g(t) > 0$. Introduce*

$$\gamma = \frac{\int_0^\infty b(s)\,ds}{\int_0^\infty s a(s)\,ds}.$$

Then $0 < \liminf_{t \to \infty} g(t) t^{-\gamma} \leq \sup_t g(t) t^{-\gamma} < \infty$ holds.

Theorem 3. *Let g be the solution of equation (4). In addition to the conditions of Theorem 2 suppose that g is decreasing. Then $g(t) t^{-\gamma} \to C$ holds for some $0 < C < \infty$ as $t \to \infty$.*

3. THE DISCRETE CASE: PROOF OF THEOREM 1.

3.1. **Preliminaries.** We may assume that $\sum_{n=1}^\infty a_n = 1$ and $q = 1$.

Condition (r3) implies that q exists, and $q < z$. Define $\tilde{x}_n = q^n x_n$, $\tilde{a}_j = q^j a_j$, $\tilde{b}_j = q^j b_j$, $\tilde{c}_{n,j} = q^j c_{n,j}$, $\tilde{r}_n = q^n r_n$ for $n, j \geq 1$. We get that $\sum_{n=1}^\infty \tilde{a}_n = 1$ and

$$\tilde{x}_n = \sum_{j=1}^{n-1} \Big(\tilde{a}_j + \frac{\tilde{b}_j}{n-j} + \tilde{c}_{n,j} \Big) \tilde{x}_{n-j} + \tilde{r}_n, \quad n = 1, 2, \ldots.$$

Moreover, condition (r3) holds with $\tilde{z} = z/q$. Thus we may assume that $\sum_{n=1}^\infty a_n = 1$, and $z > 1$, indeed.

Lemma 1. *$x_n > 0$ for every n large enough.*

Proof. If $a_k > 0$ for some integer k, then for every sufficiently large n we have $w_{n,k} > 0$. To see this, note that $\lim_{n \to \infty} \frac{b_k}{n-k} = 0$, and $\lim_{n \to \infty} c_{n,k} = 0$ holds for fixed k due to condition (r3). Hence, if $x_n > 0$ and n is large enough, then $x_{n+k} > 0$. This implies that $x_{n+\ell k} > 0$ for $\ell = 1, 2, \ldots$. Due to condition (r1), every sufficiently large n is a linear combination of some values of k for which $a_k > 0$. Therefore x_n is positive for every n large enough. $\square$

We need some more notations.

Define $y_n = x_n n^{-\gamma}$ for $n \geq 1$. We have

$$(5) \qquad y_n = \sum_{j=1}^{n-1} w_{n,j} \Big(1 - \frac{j}{n} \Big)^\gamma y_{n-j} + r_n n^{-\gamma}, \quad n = 1, 2, \ldots.$$

From the Taylor expansion of the function $f(x) = (1-x)^t$ for $x \geq 0$ we get that

$$|f(x) - 1 + tx| = \left| \tfrac{t(t-1)}{2} x^2 (1-\theta x)^{t-2} \right| \leq \tfrac{|t(t-1)|}{2} x^2 e^{-\theta x(t-2)}$$
$$\leq \tfrac{|t(t-1)|}{2} x^2 e^{x|t-2|} \leq \tfrac{|t(t-1)|}{2} x^2 e^{n\varepsilon x},$$

where $0 \leq \theta \leq 1$, and $\varepsilon > 0$ is so small that $e^{2\varepsilon} < z$ holds with z of condition (r3), while n is so large that $n\varepsilon > |t-2|$ holds.

Therefore

$$(6) \qquad \left(1 - \frac{j}{n}\right)^\gamma = 1 - \frac{\gamma j}{n} + R_{n,j},$$

where

$$(7) \qquad |R_{n,j}| \leq \frac{|\gamma(\gamma-1)|}{2} \frac{j^2}{n^2} e^{j\varepsilon}$$

holds uniformly in j for all n large enough, say $n \geq L$. Assuming that the coefficients satisfy equation (3) we get that

$$(8)$$
$$w_{n,j}\left(1 - \frac{j}{n}\right)^\gamma = a_j + \frac{1}{n}\left(b_j - \gamma j a_j\right) - \frac{1}{n^2}\gamma j b_j + w_{n,j} R_{n,j} + c_{n,j}\left(1 - \frac{\gamma j}{n}\right).$$

3.2. **Boundedness of** (y_n). Our next goal is to prove that the sequence (y_n) is bounded from above, and its limes inferior is positive. Before doing so we prove another lemma.

Lemma 2. *For every positive integer k we have*

$$(9) \qquad \sum_{n=k}^{\infty} \left| \sum_{j=1}^{n-k} w_{n,j}\left(1 - \frac{j}{n}\right)^\gamma - a_j \right| < \infty.$$

Proof. Using equation (8) we obtain that

$$\sum_{n=k}^{\infty} \left| \sum_{j=1}^{n-k} w_{n,j}\left(1 - \frac{j}{n}\right)^\gamma - a_j \right| = \sum_{n=k}^{\infty} \left| \frac{1}{n} \sum_{j=1}^{n-k} (b_j - \gamma j a_j) - \frac{\gamma}{n^2} \sum_{j=1}^{n-k} j b_j + \right.$$
$$\left. + \sum_{j=1}^{n-k} w_{n,j} R_{n,j} + \sum_{j=1}^{n-k} c_{n,j}\left(1 - \frac{\gamma j}{n}\right) \right|$$
$$\leq \sum_{n=k}^{\infty} \left(\frac{1}{n} \sum_{j=n-k+1}^{\infty} |\gamma j a_j - b_j| + \frac{|\gamma|}{n^2} \sum_{j=1}^{n-1} j|b_j| + \right.$$
$$\left. + \sum_{j=1}^{n-1} |w_{n,j} R_{n,j}| + (1 + |\gamma|) \sum_{j=1}^{n-1} |c_{n,j}| \right).$$

For the first term we used that $\sum_{j=1}^{\infty}(b_j - \gamma j a_j) = 0$ holds by the definition of γ.

Let us divide the sum into four parts and examine them separately. By condition (r3) for the first two we have that

$$\sum_{n=k}^{\infty} \frac{1}{n} \sum_{j=n-k+1}^{\infty} |\gamma j a_j - b_j| \le \sum_{n=k}^{\infty} \sum_{j=n-k+1}^{\infty} \left(|\gamma| j a_j + |b_j| \right)$$

$$= |\gamma| \sum_{j=1}^{\infty} j^2 a_j + \sum_{j=1}^{\infty} j |b_j| < \infty;$$

$$\sum_{n=1}^{\infty} \frac{|\gamma|}{n^2} \sum_{j=1}^{n-1} j |b_j| \le |\gamma| \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{j=1}^{\infty} j |b_j| < \infty.$$

Moreover, using (7), we obtain that

$$\sum_{n=L}^{\infty} \sum_{j=1}^{n-1} |w_{n,j} R_{n,j}| \le \frac{|\gamma(\gamma-1)|}{2} \sum_{n=L}^{\infty} \sum_{j=1}^{n-1} \left(a_j + |b_j| + |c_{n,j}| \right) \frac{j^2}{n^2} e^{j\varepsilon}$$

$$\le \frac{|\gamma(\gamma-1)|}{2} \left[\sum_{j=1}^{\infty} \sum_{n=j+1}^{\infty} \left(a_j + |b_j| \right) \frac{j^2}{n^2} e^{j\varepsilon} + \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} |c_{n,j}| z^j \right] < \infty$$

by condition (r3) and the choice of ε.

Similarly,

$$\sum_{n=1}^{\infty} \sum_{j=1}^{n-1} |c_{n,j}| \le \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} |c_{n,j}| z^j < \infty$$

also holds. Thus we have proved (9). $\qquad\square$

Lemma 3. (y_n) *is bounded from above.*

Proof. Let $z_n = \max\{1, y_1, \ldots, y_n\}$. Then $y_n \le z_n$, z_n is increasing, and

$$z_n \le z_{n-1} \, \max\left\{ 1, \, \sum_{j=1}^{n-1} w_{n,j} \left(1 - \frac{j}{n} \right)^{\gamma} + r_n n^{-\gamma} \right\}$$

$$\le z_{n-1} \left(1 + \left| \sum_{j=1}^{n-1} w_{n,j} \left(1 - \frac{j}{n} \right)^{\gamma} - a_j \right| + r_n n^{-\gamma} \right)$$

$$= z_{n-1}(1 + s_n),$$

where $s_n = \left| \sum_{j=1}^{n-1} w_{n,j} \left(1 - \frac{j}{n} \right)^{\gamma} - a_j \right| + r_n n^{-\gamma}$.

Iterating this we obtain that $\sup_{n \ge 1} z_n \le \prod_{n=1}^{\infty} (1 + s_n)$. In order to show that this quantity is finite it is sufficient to prove that $\sum_{n=1}^{\infty} s_n < \infty$ holds. The latter is implied by Lemma 2 with $k = 1$, and by the fact that

$$\sum_{n=1}^{\infty} r_n n^{-\gamma} = O\left(\sum_{n=1}^{\infty} r_n z^n \right) < \infty.$$

Thus the sequence (y_n) is bounded from above. $\square$

Lemma 4. $\liminf_{n\to\infty} y_n > 0$.

Proof. This is similar to the upper bound, therefore we only outline the proof, omitting the details.

Based on Lemma 1, suppose that $x_n > 0$ for all $n \geq N$. By condition (r2) and the definition of y_n we have that

$$y_n \geq \sum_{j=1}^{n-N} w_{n,j}\left(1 - \frac{j}{n}\right)^\gamma y_{n-j}.$$

We obtain that for $z_n = \min_{N \leq j \leq n} y_j$ the following inequality holds.

$$
\begin{aligned}
z_n &\geq z_{n-1} \min\left\{1, \sum_{j=1}^{n-N} w_{n,j}\left(1 - \frac{j}{n}\right)^\gamma\right\} \\
&\geq z_{n-1}\left(\sum_{j=1}^{n-N} a_j - \left|\sum_{j=1}^{n-N} w_{n,j}\left(1 - \frac{j}{n}\right)^\gamma - a_j\right|\right) \\
&= z_{n-1}\left(1 - \sum_{j=n-N+1}^{\infty} a_j - \left|\sum_{j=1}^{n-N} w_{n,j}\left(1 - \frac{j}{n}\right)^\gamma - a_j\right|\right) \\
&= z_{n-1}(1 - s_n),
\end{aligned}
$$

where

$$s_n = \sum_{j=n-N+1}^{\infty} a_j + \left|\sum_{j=1}^{n-N} w_{n,j}\left(1 - \frac{j}{n}\right)^\gamma - a_j\right|.$$

This implies that $\inf_{n \geq N} y_n \geq \lim_{n\to\infty} z_n \geq z_N \prod_{n=N+1}^{\infty}(1 - s_n)$. For the proof of the positivity of the right-hand side we need to show that $\sum_{n=N+1}^{\infty} s_n < \infty$. This is a consequence of Lemma 2 with $k = N$, and the following estimation.

$$\sum_{n=N+1}^{\infty} \sum_{j=n-N+1}^{\infty} a_j \leq \sum_{j=1}^{\infty} j a_j < \infty.$$

We conclude that $\inf_{n \geq N} y_n > 0$, and hence $\liminf_{n\to\infty} y_n > 0$. $\square$

3.3. **Final step.** The remaining part of the proof is similar to the proof of the discrete renewal theorem.

Equation (8) implies that

$$(10) \quad y_n = \sum_{j=1}^{n-1}\left(a_j + \frac{b_j - \gamma j a_j}{n-j} - \frac{(b_j - \gamma j a_j)j}{n(n-j)} - \frac{1}{n^2}\gamma j b_j + \right.$$
$$\left. + w_{n,j}R_{n,j} + c_{n,j}\left(1 - \frac{\gamma j}{n}\right)\right)y_{n-j} + r_n n^{-\gamma}.$$

Fix a positive integer N. Then we obtain from (10) by summation that

$$(11) \quad \sum_{n=1}^{N}y_n = \sum_{n=1}^{N}\sum_{j=1}^{n-1}\left(a_j + \frac{b_j - \gamma j a_j}{n-j}\right)y_{n-j} + u_N$$

with an appropriately chosen sequence (u_N). Here u_N is convergent as $N \to \infty$; let u denote the limit. In order to show this, since (y_n) is bounded, it is sufficient to prove that

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n-1}\left(\frac{|b_j - \gamma j a_j|j}{n(n-j)} + \frac{1}{n^2}|\gamma b_j|j + |w_{n,j}R_{n,j}| + |c_{n,j}|\left(1 + |\gamma|\right)\right) < \infty;$$

$$\sum_{n=1}^{\infty}r_n n^{-\gamma} < \infty.$$

We have almost done it before; the only thing left is to show the convergence for the first term in the double sum. In this case

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n-1}\frac{|b_j - \gamma j a_j|j}{n(n-j)} \leq \sum_{n=1}^{\infty}\sum_{j=1}^{n-1}\frac{j|b_j| + |\gamma|j^2 a_j}{n(n-j)}$$
$$= \sum_{j=1}^{\infty}\left(j^2|b_j| + |\gamma|j^3 a_j\right)\sum_{n=j+1}^{\infty}\frac{1}{n(n-j)}$$
$$\leq \sum_{j=1}^{\infty}\left(j^2|b_j| + |\gamma|j^3 a_j\right)\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty.$$

Introduce variable $m = n - j$ instead of n in equation (11). Then

$$\sum_{n=1}^{N}y_n = \sum_{j=1}^{N-1}\sum_{n=j+1}^{N}\left(a_j + \frac{b_j - \gamma j a_j}{n-j}\right)y_{n-j} + u_N$$
$$= \sum_{j=1}^{N-1}\sum_{m=1}^{N-j}\left(a_j + \frac{b_j - \gamma j a_j}{m}\right)y_m + u_N$$
$$= \sum_{m=1}^{N-1}y_m\sum_{j=1}^{N-m}\left(a_j + \frac{b_j - \gamma j a_j}{m}\right) + u_N.$$

Since
$$\sum_{j=1}^{\infty}\left(a_j + \frac{b_j - \gamma j a_j}{m}\right) = 1,$$
we have
$$\sum_{n=1}^{N} y_n = \sum_{m=1}^{N} y_m\left(1 - \sum_{j=N-m+1}^{\infty}\left(a_j + \frac{b_j - \gamma j a_j}{m}\right)\right) + u_N.$$
These imply that
$$\sum_{m=0}^{N-1} y_{N-m} \sum_{j=m+1}^{\infty} a_j = \sum_{m=1}^{N} y_m \sum_{j=N-m+1}^{\infty} a_j$$
$$= u_N - \sum_{m=1}^{N} y_m \sum_{j=N-m+1}^{\infty} \frac{b_j - \gamma j a_j}{m}.$$
The second term on the right-hand side converges to 0 as $N \to \infty$, because
$$\sum_{m=1}^{N} y_m \sum_{j=N-m+1}^{\infty} \frac{|b_j - \gamma j a_j|}{m} \leq (\sup_n y_n) \sum_{m=1}^{N} \frac{1}{m} \sum_{j=N-m+1}^{\infty} \left(|b_j| + |\gamma| j a_j\right).$$
The sum inside is estimated in the following way. Let $\varepsilon > 0$ such that $e^\varepsilon < z$ holds. Then
$$\sum_{j=N-m+1}^{\infty} \left(|b_j| + |\gamma| j a_j\right) \leq K\, e^{-(N-m)\varepsilon},$$
where
$$K = \sum_{j=1}^{\infty} (|b_j| + |\gamma| j a_j) e^{j\varepsilon} < \infty.$$
Now, using notation $M = \left[\sqrt{N}\right]$, we get that
$$\sum_{m=1}^{N} \frac{K}{m} e^{-(N-m)\varepsilon} \leq \sum_{m=1}^{N-M} \frac{K}{m} e^{-(N-m)\varepsilon} + \sum_{m=N-M+1}^{N} \frac{K}{m} e^{-(N-m)\varepsilon}$$
$$\leq NKe^{-M\varepsilon} + \frac{MK}{N-M},$$
which tends to 0 as $N \to \infty$. We conclude that
$$(12) \qquad \sum_{m=0}^{N-1} y_{N-m} \sum_{j=m+1}^{\infty} a_j \to u \quad (N \to \infty).$$

Modifying the proof of Lemma 2, namely, using equation (8), condition (r3) and the fact that the sequence of arithmetic means converges

to zero if the original sequence is nonnegative and converges to zero, it is easy to see that

$$\sum_{j=1}^{n-1} \left| w_{n,j}\left(1 - \frac{j}{n}\right)^{\gamma} - a_j \right| \to 0 \quad (n \to \infty).$$

This and (5), together with the boundedness of (y_n) imply that

$$(13) \qquad\qquad y_n - \sum_{j=1}^{n-1} a_j y_{n-j} \to 0,$$

as $n \to \infty$.

From now on the argument is the usual one.

Let (n_k) be a subsequence of the natural numbers that satisfies

$$\lim_{k\to\infty} y_{n_k} = \limsup_{n\to\infty} y_n =: \overline{y}.$$

From (13) we get that for all $\ell \le M$ the following estimation holds.

$$\overline{y} = \lim_{k\to\infty} y_{n_k} \le a_\ell \liminf_{k\to\infty} y_{(n_k - \ell)} + \limsup_{k\to\infty} \sum_{j < n_k,\, j \ne \ell} a_j y_{(n_k - j)}$$

$$\le a_\ell \liminf_{k\to\infty} y_{(n_k - \ell)} + \sum_{j \ne \ell,\, j \le M} a_j \limsup_{k\to\infty} y_{(n_k - j)} + \left(\sup_n y_n\right) \sum_{j=M+1}^{\infty} a_j$$

$$\le a_\ell \liminf_{k\to\infty} y_{(n_k - \ell)} + (1 - a_\ell)\overline{y} + \left(\sup_n y_n\right) \sum_{j=M+1}^{\infty} a_j.$$

Since M may be arbitrarily large, this immediately implies that $\lim_{k\to\infty} y_{(n_k - \ell)} = \overline{y}$, for all $a_\ell > 0$. By iteration we obtain that

$$\lim_{k\to\infty} y_{(n_k - \ell_1 - \ldots - \ell_i)} = \overline{y},$$

for all positive $a_{\ell_1}, \ldots, a_{\ell_i}$. By condition (r1) for all sufficiently large m we have $\lim_{k\to\infty} y_{n_k - m} = \overline{y}$. Modifying the subsequence we may assume that this holds for all $m = 0, 1, \ldots$. Hence choosing $N = n_k$ in (12) we can see that

$$\overline{y} \sum_{m=0}^{\infty} \sum_{j=m+1}^{\infty} a_j = u.$$

For $\underline{y} = \liminf_{n\to\infty} y_n$ the same argument shows that

$$\underline{y} \sum_{m=0}^{\infty} \sum_{j=m+1}^{\infty} a_j = u.$$

Hence $\overline{y} = \underline{y}$, that is, the limit $\lim_{n\to\infty} y_n = C$ exists. We have already proved that this is finite and positive. This implies Theorem 1. $\qquad\square$

Remark 3. It is well known that if $y_n = \sum_{j=1}^{n-1} a_j y_{n-j}$ holds for all n, instead of in (13), then y_n is convergent. On the other hand, (13) is not yet sufficient for the convergence of (y_n). For example, let $y_n = 2 + \sin(\log(1+n))$. Then

$$|y_n - y_{n-j}| \le \log(1+n) - \log(1+n-j) \le \frac{j}{1+n-j}\,,$$

hence

$$\left| y_n - \sum_{j=1}^{n-1} a_j y_{n-j} \right| \le y_n \sum_{j=n}^{\infty} a_j + \sum_{j=1}^{n-1} |y_n - y_{n-j}|\, a_j$$

$$\le 3 \sum_{j=n}^{\infty} a_j + 3 \sum_{j=1}^{n-1} \frac{ja_j}{1+n-j}$$

$$\le 3 \sum_{j=n}^{\infty} a_j + 3 \sum_{j=1}^{M-1} \frac{ja_j}{n-M} + 3 \sum_{j=M}^{n-1} ja_j$$

$$\le 3 \sum_{j=n}^{\infty} a_j + \frac{3}{n-M} \sum_{j=1}^{\infty} ja_j + 3 \sum_{j=M}^{\infty} ja_j.$$

This converges to zero with $M = n/2$, but y_n does not converge.

Remark 4. If $w_{k,i} = a_i$, then $x_k \to C$ by the arithmetic version of the renewal theorem. The following example shows that a remainder, though converging to 0, may change this. Let (a_i) be arbitrary, $x_k = 2 + \sin(\log(k+1))$, and

$$w_{k,i} = a_i + \left(x_k - \sum_{j=1}^{k-1} x_{k-j} a_j \right) \left(\sum_{j=1}^{k-1} x_j \right)^{-1}.$$

Then $w_{k,i} = a_i + o(1)$, and $x_k = \sum_{i=1}^{k-1} x_{k-i} w_{k,i} + 2\delta_{k,1}$.

4. The continuous case: Proof of Theorem 2.

4.1. Preliminaries.

Lemma 5. *Under the conditions of Theorem 2 we have $g(t) > 0$ for every t large enough.*

Proof. Choose $0 < s_0$ and $\delta > 0$ such that the set $S' = \{s \in (0, s_0) : a(s) > 3\delta\}$ has positive Lebesgue measure.

Let $\ell(s) = \sup\{t \ge s : |c_{t,s}| \ge \delta\}$. Then $\ell(s) < \infty$ for a.e. $s > 0$ by condition (i4). Though ℓ is not necessarily Borel measurable, yet it is Lebesgue measurable by the measurable projection theorem, for the superlevel set $\{\ell > K\}$ is just the projection of the two dimensional measurable set $\{(s,t) : 0 < s \le t,\ K < t,\ |c_{t,s}| \ge \delta\}$ onto the first

coordinate. Hence $U_t = \{\ell < t\}$, $t > 0$ is an increasing family of Lebesgue measurable sets, and the Lebesgue measure of $(0, s_0) \setminus U_t$ tends to 0 as $t \to \infty$. The same holds for the sequence $V_t = \{s \in (0, t] : |b(s)| \leq \delta(t + d)\}$. Thus we can find a threshold $T \geq s_0$ such that the Lebesgue measure of $S = S' \cap U_T \cap V_T$ is positive. Obviously, $w_{t,s} \geq \delta$ for all $t \geq T$ and $s \in S$. By the Lebesgue density theorem we may assume that S only consists of points with density 1.

By the continuity of g there exists a whole open interval I above T where $g(t)$ is separated from zero. Let ε denote the length of I, and $\eta > 0$ the infimum of g over I. Then for $t \in S + I$ we have

$$g(t) \geq \int_0^t w_{t,s}\, g(t - s)\, ds \geq \int_{S \cap (t-I)} w_{t,s}\, g(t - s)\, ds$$
$$\geq \delta\eta\, \lambda\big(S \cap (t - I)\big) > 0,$$

where λ stands for the Lebesgue measure.

Since the set sum $S + I$ is an open set, we can iterate this procedure to obtain that g is positive everywhere on the set

$$I \cup (S + I) \cup (S + S + I) \cup (S + S + S + I) \cup \dots .$$

The proof can be completed by showing that this set contains every sufficiently large real number. In other words, if t is large enough, then it can be written in the form $t = s_1 + \dots + s_n + r$, where $s_1, \dots, s_n \in S$, $n \in \mathbb{N}$, and $0 < r < \varepsilon$. In fact, this is true for arbitrary $S \subset \mathbb{R}^+$ that has two incommensurable elements α and β (hence for every set of positive Lebesgue measure). Indeed, by the equidistribution theorem there exists positive integers k and m such that

$$k < m\frac{\alpha}{\beta} < k + \frac{\varepsilon}{\beta},$$

that is, $k\beta < m\alpha$, and their distance is less than ε. Consequently, in the finite sequence

$$nk\beta < (n-1)k\beta + m\alpha < (n-2)k\beta + 2m\alpha < \dots < nm\alpha$$

the distance between neighbouring terms is less than ε. If n is large enough, then $(n+1)k\beta < nm\alpha$, i.e., the largest term of the sequence above is bigger than the smallest term of the next sequence. Thus every $t \geq nk\beta$ is sufficiently close to a positive linear combination of α and β. $\qquad\square$

Let us introduce the notation

$$H(t) = g(t)\,(t + d)^{-\gamma} \qquad (t \geq 0).$$

From (4) we obtain the following integral equation for H.

$$(14) \qquad H(t) = \int_0^t w_{t,s} \left(\frac{t - s + d}{t + d}\right)^\gamma H(t - s)\, ds + r(t)\,(t + d)^{-\gamma}$$

for $t > 0$, and $H(0) = d^{-\gamma}$.

Let us choose ε in a similar way as we did in the discrete case. Namely, we have $e^{2\varepsilon} < z$ with z of condition (i5). In what follows equations (15), (16), and (17) correspond to (6), (7), and (8), resp. Firstly,

$$(15) \qquad \left(\frac{t - s + d}{t + d} \right)^{\gamma} = \left(1 - \frac{s}{t + d} \right)^{\gamma} = 1 - \frac{\gamma s}{t + d} + R_{t,s},$$

where

$$(16) \qquad |R_{t,s}| \leq \frac{|\gamma(\gamma - 1)|}{2} \frac{s^2}{t^2} e^{s\varepsilon},$$

if t is large enough, say $t \geq L$. Finally, from the decomposition of $w_{t,s}$ we get that

$$(17) \quad w_{t,s} \left(1 - \frac{s}{t + d} \right)^{\gamma}$$
$$= a(s) + \frac{b(s) - \gamma s a(s)}{t + d} - \frac{\gamma s b(s)}{(t + d)^2} + w_{t,s} R_{t,s} + c_{t,s} \left(1 - \frac{\gamma s}{t + d} \right).$$

4.2. **Boundedness of H.** The method of proof is discretization; in this way all we need to do is similar to what we did in the discrete case.

Before proving boundedness we need another lemma.

Lemma 6. *For every fixed $T \geq 0$ the function*

$$A(t) = \left| \int_0^{t-T} \left(w_{t,s} \left(1 - \frac{s}{t + d} \right)^{\gamma} - a(s) \right) ds \right|$$

is directly Riemann integrable on $[T, \infty)$.

Proof. Fix $\tau > 0$. Then

$$(18) \qquad \sum_{n=1}^{\infty} \sup_{n\tau \leq \theta \leq (n+1)\tau} \int_0^{\theta} |c_{\theta,s}| z^s ds < \infty,$$

according to condition (i6). Now we prove that

$$(19) \qquad \sum_{n=\left\lceil \frac{T}{\tau} \right\rceil}^{\infty} \sup_{n\tau \leq \theta \leq (n+1)\tau} A(\theta) = \sum_{n=\left\lceil \frac{T}{\tau} \right\rceil}^{\infty} \sup_{0 \leq \theta \leq \tau} A(n\tau + \theta) < \infty.$$

From the definition of γ it follows that $\int_0^\infty \big(b(s) - \gamma s a(s)\big) ds = 0$. Using this and equation (17) we obtain that

$$
\begin{aligned}
A(n\tau + \theta) &= \left| \int_0^{n\tau+\theta-T} \left(w_{n\tau+\theta,s} \left(1 - \frac{s}{n\tau+\theta+d} \right)^\gamma - a(s) \right) ds \right| \\
&= \left| \int_0^{n\tau+\theta-T} \frac{b(s) - \gamma s a(s)}{n\tau+\theta+d}\, ds - \frac{\gamma}{(n\tau+\theta+d)^2} \int_0^{n\tau+\theta-T} s b(s)\, ds \right. \\
&\quad \left. + \int_0^{n\tau+\theta-T} w_{n\tau+\theta,s} R_{n\tau+\theta,s}\, ds + \int_0^{n\tau+\theta-T} c_{n\tau+\theta,s} \left(1 - \frac{\gamma s}{n\tau+\theta+d} \right) ds \right| \\
&\leq \int_{n\tau+\theta-T}^\infty \frac{|b(s) - \gamma s a(s)|}{n\tau+\theta+d}\, ds + \frac{|\gamma|}{(n\tau+\theta+d)^2} \int_0^{n\tau+\theta} s\,|b(s)|\, ds \\
&\quad + \int_0^{n\tau+\theta} |w_{n\tau+\theta,s} R_{n\tau+\theta,s}|\, ds + \big(1 + |\gamma|\big) \int_0^{n\tau+\theta} |c_{n\tau+\theta,s}|\, ds
\end{aligned}
$$

holds for all $\theta > 0$.

We treat the four integrals in the right-hand side separately again. For the first term we get that

$$
\begin{aligned}
\sum_{n=\left\lceil \frac{T}{\tau} \right\rceil}^\infty \sup_{0 \leq \theta \leq \tau} \frac{1}{n\tau+\theta+d} \int_{n\tau+\theta-T}^\infty |b(s) - \gamma s a(s)|\, ds & \\
&\leq \sum_{n=\left\lceil \frac{T}{\tau} \right\rceil}^\infty \frac{1}{n\tau+d} \int_{n\tau-T}^\infty |b(s) - \gamma s a(s)|\, ds \\
&\leq \sum_{n=1}^\infty \frac{n}{n\tau+d+T} \int_{n\tau}^{(n+1)\tau} |b(s) - \gamma s a(s)|\, ds \\
&\leq \frac{1}{\tau} \int_0^\infty \big(|\gamma| s a(s) + |b(s)| \big) ds,
\end{aligned}
$$

which is finite by condition (i5).

The sum of the second terms is also finite by condition (i5).

$$
\begin{aligned}
\sum_{n=1}^\infty \sup_{0 \leq \theta \leq \tau} \frac{|\gamma|}{(n\tau+\theta+d)^2} \int_0^{n\tau+\theta} s\,|b(s)|\, ds & \\
&\leq \frac{|\gamma|}{\tau^2} \sum_{n=1}^\infty \frac{1}{n^2} \int_0^\infty s\,|b(s)|\, ds < \infty.
\end{aligned}
$$

By inequality (16), condition (i5), and the choice of ε in the preliminaries, we obtain the following estimation for the sum of the third

terms.

$$\sum_{n=\left\lceil \frac{L}{\tau} \right\rceil}^{\infty} \sup_{0 \le \theta \le \tau} \int_0^{n\tau+\theta} |w_{n\tau+\theta,s} R_{n\tau+\theta,s}| \, ds$$

$$\le \sum_{n=\left\lceil \frac{L}{\tau} \right\rceil}^{\infty} \sup_{0 \le \theta \le \tau} \int_0^{n\tau+\theta} \left(a(s) + \frac{|b(s)|}{d} + |c_{n\tau+\theta,s}| \right) \frac{|\gamma(\gamma-1)| \, s^2}{2(n\tau+\theta)^2} e^{s\varepsilon}$$

$$\le \frac{|\gamma(\gamma-1)|}{2} \sum_{n=1}^{\infty} \int_0^{(n+1)\tau} \left(a(s) + \frac{|b(s)|}{d} \right) \frac{s^2}{\tau^2 n^2} e^{s\varepsilon} \, ds$$

$$+ \frac{|\gamma(\gamma-1)|}{2} \sum_{n=1}^{\infty} \sup_{0 \le \theta \le \tau} \int_0^{n\tau+\theta} |c_{n\tau+\theta,s}| \, z^s \, ds.$$

In the right-hand side the last sum is finite by (18). In the first sum the integrand is nonnegative, hence by Fubini's theorem we obtain that

$$\sum_{n=1}^{\infty} \int_0^{(n+1)\tau} \left(a(s) + \frac{|b(s)|}{d} \right) \frac{s^2}{\tau^2 n^2} e^{s\varepsilon} \, ds$$

$$\le \frac{1}{\tau^2} \int_0^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) \left(a(s) + \frac{|b(s)|}{d} \right) s^2 e^{s\varepsilon} \, ds < \infty,$$

by the choice of ε. Thus the sum of the third terms is finite, too.

Finally, for the sum of the fourth terms we clearly have

$$\sum_{n=1}^{\infty} \sup_{0 \le \theta \le \tau} \int_0^{n\tau+\theta} |c_{n\tau+\theta,s}| \, ds \le \sum_{n=1}^{\infty} \sup_{0 \le \theta \le \tau} z^{n\tau+\theta} \int_0^{n\tau+\theta} |c_{n\tau+\theta,s}| \, ds < \infty.$$

Putting these together we obtain that all four parts of A give finite sums, hence (19) holds. The nonnegativity and integrability of A is clear, for it is a continuous function of t. Thus the proof of the lemma is completed. $\qquad\square$

Lemma 7. $H(t)$ *is bounded from above.*

We define $Z(t) = \max\{1, H(s) : 0 \le s \le t\}$ for $t \ge 0$. This is finite, because H, as well as g, is continuous.

First we give an upper bound for $\sup_{0 < \theta \le \tau} H(t+\theta)$, where t and τ are fixed positive numbers. Introduce

$$w_{t,s}^* = w_{t,s} \left(1 - \frac{s}{t+d} \right)^{\gamma} \qquad 0 < t, \ 0 \le s \le t.$$

Using the nonnegativity of w and r, equation (14), and the definition of Z, we get that

$$(20) \quad H(t+\theta) = \int_0^\theta w_{t+\theta,s}^* H(t+\theta-s)\,ds$$

$$+ \int_\theta^{t+\theta} w_{t+\theta,s}^* H(t+\theta-s)\,ds + r(t+\theta)(t+\theta+d)^{-\gamma}$$

$$\leq \int_0^\theta w_{t+\theta,s}^*\,ds\,Z(t+\theta) + \left[\int_\theta^{t+\theta} w_{t+\theta,s}^*\,ds + r(t+\theta)(t+\theta+d)^{-\gamma}\right] Z(t)$$

$$= \int_0^\theta w_{t+\theta,s}^*\,ds\,[Z(t+\theta)-Z(t)]$$

$$+ \left[\int_0^{t+\theta} w_{t+\theta,s}^*\,ds + r(t+\theta)(t+\theta+d)^{-\gamma}\right] Z(t).$$

Next we want to prove that there exists $\tau_0 > 0$, and for every τ, $0 < \tau \leq \tau_0$, a positive integer $N(\tau)$ such that

$$(21) \qquad \sup_{0\leq\theta\leq\tau} \int_0^\theta w_{n\tau+\theta,s}^*\,ds \leq \frac{1}{2},$$

provided $n > N(\tau)$.

To show this we will give an upper bound on

$$(22) \quad w_{n\tau+\theta,s}^* = \left(1 - \frac{s}{n\tau+\theta+d}\right)^\gamma \left(a(s) + \frac{b(s)}{n\tau+\theta+d} + c_{n\tau+\theta,s}\right).$$

We clearly have

$$\left(1 - \frac{s}{n\tau+\theta+d}\right)^\gamma \leq \left(1 + \frac{\theta}{d}\right)^{|\gamma|} \leq \exp\left(\frac{\theta|\gamma|}{d}\right),$$

and

$$a(s) + \frac{b(s)}{n\tau+\theta+d} \leq a(s) + \frac{|b(s)|}{d}.$$

Hence, for $0 \leq \theta \leq \tau$ we can write

$$\int_0^\theta w_{n\tau+\theta,s}^*\,ds \leq \exp\left(\frac{\theta|\gamma|}{d}\right)\left[\int_0^\theta \left(a(s) + \frac{|b(s)|}{d}\right)ds + \int_0^\theta |c_{n\tau+\theta,s}|\,ds\right]$$

$$\leq \exp\left(\frac{\tau|\gamma|}{d}\right)\int_0^\tau \left(a(s) + \frac{|b(s)|}{d}\right)ds + z^{n\tau+\theta}\int_0^\theta |c_{n\tau+\theta,s}|\,ds,$$

if n is large enough, namely, $n \geq |\gamma|/2d\varepsilon$ will do. The first term in the right-hand side can be arbitrarily small if τ is fixed small enough. As to the second term, it can be estimated in the following way.

$$\sup_{0\leq\theta\leq\tau} z^{n\tau+\theta}\int_0^\theta |c_{n\tau+\theta,s}|\,ds \leq \sup_{n\tau\leq\theta\leq(n+1)\tau} z^\theta \int_0^\theta |c_{\theta,s}|\,ds,$$

which tends to 0 as $n \to \infty$ by condition (i6). Thus (21) is satisfied if n is greater than a certain threshold $N(\tau)$.

For any $0 < \tau \leq \tau_0$ and $t = n\tau$, $n > N(\tau)$ inequality (20) implies that

$$\sup_{0 < \theta \leq \tau} H(t+\theta) \leq \frac{1}{2}\left[Z(t+\tau) - Z(t)\right] +$$
$$+ \sup_{0 < \theta \leq \tau} \left[\int_0^t w_{t+\theta,s}^* ds + r(t+\theta+d)(t+\theta+d)^{-\gamma}\right] Z(t).$$

Here we use that Z is nonnegative and increasing by definition.

We clearly have $Z(t+\tau) = \max\left\{Z(t), \sup_{0 < \theta \leq \tau} H(t+\theta)\right\}$. Therefore we obtain that

$$Z(t+\tau) \leq \max\left\{Z(t), \quad \frac{1}{2}\left[Z(t+\tau) - Z(t)\right] + \right.$$
$$\left. + \sup_{0 < \theta \leq \tau} \left[\int_0^t w_{t+\theta,s}^* ds + r(t+\theta)(t+\theta+d)^{-\gamma}\right] Z(t)\right\},$$

from which it follows that

$$Z(t+\tau) - Z(t) \leq \left(\frac{1}{2}\left[Z(t+\tau) - Z(t)\right] + \right.$$
$$\left. + \sup_{0 < \theta \leq \tau} \left[\int_0^t w_{t+\theta,s}^* ds - 1 + r(t+\theta)(t+\theta+d)^{-\gamma}\right] Z(t)\right)^+;$$

where x^+ denotes $\max(x, 0)$, as usual. Hence

$$Z(t+\tau) - Z(t)$$
$$\leq 2\left(\sup_{0 < \theta \leq \tau} \left[\int_0^t w_{t+\theta,s}^* ds - 1 + r(t+\theta)(t+\theta+d)^{-\gamma}\right]\right)^+ Z(t).$$

We continue with deriving an upper bound for the right-hand side. Since a is a probability density function, we have

$$\sup_{0 < \theta \leq \tau} \left[\int_0^{t+\theta} w_{t+\theta,s}^* ds - 1\right] \leq \sup_{0 < \theta \leq \tau} \left[\int_0^{t+\theta} w_{t+\theta,s}^* ds - \int_0^{t+\theta} a(s)\, ds\right]$$
$$\leq \sup_{0 < \theta \leq \tau} \left|\int_0^{t+\theta} \left(w_{t+\theta,s}^* - a(s)\right) ds\right|.$$

Therefore

$$Z(t+\tau) - Z(t) \leq Z(t) \left(\sup_{0<\theta\leq\tau} \left| \int_0^{t+\theta} \left(w_{t+\theta,s}^* - a(s) \right) ds \right| + \right.$$

$$\left. + \sup_{0<\theta\leq\tau} r(t+\theta)(t+\theta+d)^{-\gamma} \right)$$

for all $0 < \tau \leq \tau_0$, $t = n\tau$, $n \geq N(\tau)$.

Similarly to Lemma 3 of the discrete case, for the boundedness of $Z(n\tau)$ from above it suffices to prove that

$$(23) \quad \sum_{n=1}^{\infty} \sup_{0<\theta\leq\tau} \left| \int_0^{n\tau+\theta} \left(w_{n\tau+\theta,s}^* - a(s) \right) ds \right| +$$

$$+ \sum_{n=1}^{\infty} \sup_{0\leq\theta\leq\tau} r(n\tau+\theta)(n\tau+\theta+d)^{-\gamma} < \infty.$$

Lemma 6 with $T = 0$ implies that the first sum is finite. Since by condition (i6) $r(t)z^t$ is directly Riemann integrable, it follows that the second sum is also finite. Thus we conclude that the sequence

$$Z(n\tau) = \max\{1, \, H(s) : 0 \leq s \leq n\tau\}$$

is bounded from above if τ is small enough. Hence the function H is also bounded from above. □

Lemma 8. $\liminf_{t\to\infty} H(t) > 0$.

Proof. Like in the discrete case, we omit the details that are straightforward modifications of the previous lemma, and only give a sketch of the proof.

Based on Lemma 5, we can suppose that $H(t) > 0$ for all $t \geq T$. This time define

$$Z(t) = \min\{H(s) : T \leq s \leq t\},$$

for $t \geq T$.

Let us derive a lower bound for $H(t+\theta)$, where $t > T$ and $\theta > 0$.

$$H(t+\theta) = \int_0^{t+\theta} w_{t+\theta,s}^* H(t+\theta-s) \, ds + r(t+\theta)(t+\theta+d)^{-\gamma}$$

$$\geq \int_0^{\theta} w_{t+\theta,s}^* H(t+\theta-s) \, ds + \int_{\theta}^{t-T+\theta} w_{t+\theta,s}^* H(t+\theta-s) \, ds$$

$$\geq Z(t+\theta) \int_0^{\theta} w_{t+\theta,s}^* \, ds + Z(t) \int_{\theta}^{t-T+\theta} w_{t+\theta,s}^* \, ds$$

$$= [Z(t+\theta) - Z(t)] \int_0^{\theta} w_{t+\theta,s}^* \, ds + Z(t) \int_0^{t-T+\theta} w_{t+\theta,s}^* \, ds.$$

Now Z is decreasing. Applying (21) we obtain that

$$H(t + \theta) \geq \frac{1}{2} \left[Z(t + \theta) - Z(t) \right] + Z(t) \int_0^{t-T+\theta} w^*_{t+\theta,s} \, ds$$

for $0 < \theta \leq \tau$. Taking infimum, subtracting $Z(t)$ and using that a is a probability density function we get that

$$Z(t + \tau) - Z(t) \geq \min \left\{ 0, \quad \frac{1}{2} \left[Z(t + \tau) - Z(t) \right] + \right.$$
$$\left. + Z(t) \left[\inf_{0 < \theta \leq \tau} \int_0^{t-T+\theta} w^*_{t+\theta,s} \, ds - 1 \right] \right\},$$

from which it follows that

$$Z(t + \tau) - Z(t) \geq 2 \min \left\{ 0, \; Z(t) \left[\inf_{0 < \theta \leq \tau} \int_0^{t-T+\theta} w^*_{t+\theta,s} \, ds - 1 \right] \right\}$$
$$\geq -2 \left[\sup_{0 < \theta \leq \tau} \left| \int_0^{t-T+\theta} \left(w^*_{t+\theta,s} - a(s) \right) ds \right| + \int_{t-T}^{\infty} a(s) \, ds \right] Z(t).$$

Similarly to Lemma 4, in order to prove that $\lim_{n \to \infty} Z(T + n\tau) > 0$ it suffices to show that

$$\sum_{n=1}^{\infty} \int_{(n-1)\tau}^{\infty} a(s) \, ds + \sum_{n=1}^{\infty} \sup_{0 < \theta \leq \tau} \left| \int_0^{n\tau+\theta} \left(w^*_{T+n\tau+\theta,s} - a(s) \right) ds \right| < \infty.$$

For the first term we have

$$\sum_{n=1}^{\infty} \int_{(n-1)\tau}^{\infty} a(s) \, ds \leq \frac{1}{\tau} \int_0^{\infty} s a(s) \, ds < \infty$$

by condition (i5). The finiteness of the second term follows directly from Lemma 6.

Thus we proved that $\lim_{n \to \infty} Z(T + n\tau) > 0$, which immediately implies that $\liminf_{t \to \infty} H(t) > 0$, as needed. $\qquad \square$

5. THE MONOTONIC CASE: PROOF OF THEOREM 3

First note that $\gamma \leq 0$ follows from the assumption that g is decreasing.

We consider the following integral equation.

$$(24) \quad g(x) = \int_0^x g(x - u) a(u) \, du + \int_0^x g(x - u) \frac{b(u)}{x + d} \, du +$$
$$+ \int_0^x g(x - u) c_{x,u} \, du + r(x).$$

In the sequel we define the Laplace transform of an integrable function f as

$$F(s) = \lim_{y \to \infty} \int_0^y e^{-sx} f(x) \, dx$$

for $s \in \mathbb{C}$, provided the limit exists and it is finite.

Denote the Laplace transforms of functions g, a, b, r by G, A, B, R, respectively. These Laplace transforms are well defined and holomorphic on the half-plane $\mathbb{H} = \{s \in \mathbb{C} : \operatorname{Re} s >\}$; this follows from the conditions on a, b, r and Theorem 2. Moreover, A, B, and R are also holomorphic in a neighbourhood of the origin. Let f be either of the functions above, then we have

$$F'(s) = - \int_0^\infty e^{-sx} x f(x) \, dx, \quad s \in \mathbb{H}.$$

Multiplying both sides of equation (24) by $(x + d)e^{-sx}$, then integrating, and using the well known properties of the Laplace transform we obtain that

$$-G'(s) + d \cdot G(s) = - [G(s)A(s)]' +$$
$$+ d \cdot G(s)A(s) + G(s)B(s) + C(s) - R'(s) + d \cdot R(s)$$

for $s \in \mathbb{H}$, where

$$C(s) = \int_0^\infty e^{-sx}(x + d) \int_0^x g(x - u) c_{x,u} \, du \, dx.$$

This is finite and holomorphic in a neighbourhood of 0 by condition (i6) and Theorem 2.

After rearranging we have

$$G'(s) = G(s) \left[d - \frac{B(s) - A'(s)}{1 - A(s)} \right] + \frac{R'(s) - d \cdot R(s) - C(s)}{1 - A(s)} .$$

This is an inhomogeneous linear differential equation of order one for G. Restricted to the set of positive real numbers we know that the solution is unique with any condition of type $G(s_0) = t_0$, and there is an explicit formula for it. Introducing the notations

$$L(s) = d - \frac{B(s) - A'(s)}{1 - A(s)}, \quad R^*(s) = - \frac{R'(s) - d \cdot R(s) - C(s)}{1 - A(s)},$$

all solutions of the differential equation can be obtained in the form
(25)
$$G(s) = \exp\left(- \int_s^1 L(t) \, dt \right) \left[C_0 + \int_s^1 R^*(t) \exp\left(\int_t^1 L(u) \, du \right) dt \right]$$

for $s > 0$, with an appropriate constant C_0.

From the results of Theorem 2 it follows that $G(s) \to 0$ as s goes to infinity on the real line. On the other hand, $L(t) \to d > 0$ as $t \to \infty$, thus the first exponential tends to infinity as $s \to \infty$. Hence there can

exist at most one C_0 for which equation (25) is satisfied. Conditions (i5) and (i6) imply that $R^*(t) \leq C_1/t$ for some C_1; in addition, $L(t) \leq C_2 d$ also holds for $t > 1/2$. Therefore we have

$$\int_s^{+\infty} R^*(t) \exp\left(\int_t^s L(u)\, du\right) dt \leq \int_s^{+\infty} \frac{C_1}{t} \exp\left(C_2 d(s-t)\right) dt \leq \frac{C_3}{s}$$

with some constant C_3, if $s > 1/2$.

By this, setting

$$C_0 = \int_1^{+\infty} R^*(t) \exp\left(\int_t^1 L(u)\, du\right) dt,$$

which is finite, in (25) we get that

$$(26) \qquad G(s) = \int_s^{+\infty} R^*(t) \exp\left(\int_t^s L(u)\, du\right) dt,$$

for $s > 0$, and this $G(s) \to 0$ as $s \to \infty$ on the real line. Hence this is the Laplace transform of g on the set of positive numbers.

Since $G(s)$ is holomorphic on the half-plane $\mathbb{H}$, it is the unique extension of the solution given above. The right-hand side of (26) is well-defined on $\mathbb{H}$, giving a holomorphic function on $\mathbb{H}$, which extends G from the set of positive real numbers to $\mathbb{H}$. Thus we have that the Laplace transform of g is given by (26) on the whole half-plane $\mathbb{H}$.

Now we examine the behaviour of G around zero. In what follows h_1, h_2, ... will always denote functions that are holomorphic in a neighbourhood of the origin. Let us start with L. Using the Taylor expansion of the exponential function we get

$$(27) \qquad A(s) = 1 - s \int_0^\infty t a(t)\, dt + \frac{s^2}{2} \int_0^\infty t^2 a(t)\, dt + \dots, \quad s \in \mathbb{C},$$

which implies that

$$L(s) = d - \frac{B(s) - A'(s)}{1 - A(s)} = \frac{B(0) - A'(0)}{s \int_0^\infty t a(t)\, dt} + h_1(s) = -\frac{\gamma + 1}{s} + h_1(s).$$

Furthermore we have

$$\int_s^1 L(t)\, dt = (\gamma + 1) \log s + h_2(s), \quad s \in \mathbb{H}.$$

Here we chose an arbitrary holomorphic branch of the logarithm on the right half-plane $\mathbb{H}$. Once the logarithm is defined on $\mathbb{H}$, then $s^{\gamma+1}$ and $s^{-(\gamma+1)}$ are also meaningful there. Thus we obtain that

$$(28) \qquad \exp\left(\int_t^s L(u)\, du\right) = \left(\frac{t}{s}\right)^{\gamma+1} \exp\left(h_2(t) - h_2(s)\right), \quad s, t \in \mathbb{H}.$$

One can similarly derive that $sR^*(s)$ is holomorphic in a neighbourhood of 0, hence

$$(29) \qquad R^*(s) \exp\big(h_2(s)\big) = \frac{h_3(s)}{s}, \quad s \in \mathbb{H}.$$

Finally, from equation (26) we obtain that

$$(30) \quad G(s) = \int_s^{+\infty} R^*(t) \exp\left(\int_t^s L(u)\, du\right) dt$$

$$= s^{-(\gamma+1)} \exp\big(-h_2(s)\big) \int_s^{+\infty} t^{\gamma+1} R^*(t) \exp\big(h_2(t)\big)\, dt, \quad s \in \mathbb{H}.$$

Suppose first that γ is not a negative integer, and consider only positive values of s. Then, with a sufficiently small positive ε, by (29) we have

$$\int_s^\infty t^{\gamma+1} R^*(t) \exp\big(h_2(t)\big)\, dt = C_4 + \int_s^\varepsilon t^{\gamma+1} R^*(t) \exp\big(h_2(t)\big)\, dt$$

$$= C_4 + \int_s^\varepsilon t^\gamma h_3(t)\, dt$$

$$= C_4 + s^{\gamma+1} h_4(s)$$

for $s \in (0, \varepsilon)$, where C_4 is a constant. Hence

$$G(s) = \exp\big(-h_2(s)\big) \big(C_4\, s^{-(\gamma+1)} + h_4(s)\big) = h_5(s) + s^{-(\gamma+1)} h_6(s),$$

from which the kth derivative of G can be written in the following form.

$$G^{(k)}(s) = h_7(s) + s^{-(\gamma+1+k)} h_8(s).$$

Choose a positive integer k such that $0 < \gamma + k + 1$, then it follows that

$$(31) \qquad s^{\gamma+k+1} G^{(k)}(s) \to K$$

as $s \to +0$, with some finite constant K.

Before going further, we prove a similar relation for $\gamma = -k$, where k is a positive integer. In this case we have

$$\int_s^\infty t^{\gamma+1} R^*(t) \exp\big(h_2(t)\big)\, dt = C_4 + \int_s^\varepsilon t^{\gamma+1} R^*(t) \exp\big(h_2(t)\big)\, dt$$

$$= C_4 + \int_s^\varepsilon t^\gamma h_3(t)\, dt$$

$$= C_4 + s^{\gamma+1} h_4(s) + C_5 \log s,$$

where C_4 and C_5 are constants, and $s \in (0, \varepsilon)$. The term $\log s$ comes from the $(k-1)$st term of the expansion of the holomorphic function

h_3. Then

$$G(s) = \exp\big(-h_2(s)\big)\big(h_9(s) + C_5\, s^{k-1} \log s\big)$$
$$= h_{10}(s) + s^{k-1} h_{11}(s) \log s,$$

consequently, $G^{(k-1)}(s) = h_{12}(s) + h_{13}(s) \log s$, and finally

$$G^{(k)}(s) = \frac{h_{14}(s)}{s} + h_{15}(s) \log s, \quad s \in (0, \varepsilon).$$

This implies that

$$s\, G^{(k)}(s) \to K$$

as $s \to +0$, with some finite constant K. Thus, (31) remains valid for negative integer values of γ.

Now we apply Karamata's Tauberian theorem (see e.g. [7, Theorem XIII.5.2], [2, Theorem 1.7.1]). We will use the following notation. Functions v and w are asymptotically equal to each other, that is, $v(x) \sim w(x)$ as $x \to 0$ (or ∞) if v/w tends to 1 as $x \to 0$ (or ∞). $v(x) \sim 0 \cdot w(x)$ means that v/w tends to 0 as $x \to 0$ (or ∞). The latter is the same as $v = o(w)$.

Theorem A. *Let U be a non-decreasing right-continuous function on $(0, \infty)$ such that its Laplace transform $\omega(s) = \int_0^\infty e^{-sx} dU(x)$ exists for $s > 0$. If ℓ is slowly varying at infinity, $0 \le \rho < \infty$, and $0 \le c < \infty$, then each of the relations*

$$\omega(s) \sim cs^{-\rho} \ell\left(\frac{1}{s}\right) \quad as\ s \to +0$$

and

$$U(t) \sim c\, \frac{1}{\Gamma(\rho+1)}\, t^\rho \ell(t) \quad as\ t \to \infty$$

implies the other.

We apply this theorem to $U(x) = \int_0^x g(u)\, u^k du$, for which ω is constant times $G^{(k)}$. From equation (31) we get that

$$(32) \qquad \int_0^x g(u)\, u^k du \sim A_k x^{\gamma+k+1}$$

as $x \to \infty$, for some $A_k \ge 0$. Note that the constant A_k depends on k.

In order to finish the proof of Theorem 3 we need another Tauberian type theorem, giving the asymptotics of $g(u)u^k$ from the asymptotics of its integral function. We will use the monotonicity of g at this point.

We say that a function f is slowly oscillating if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $f(u) < f(x)(1+\varepsilon)$ holds for all $x < u < x(1+\delta)$ (see e.g. [8, Section 6.2], [2, Section 1.7.6,], [9, Section 17]). Using that g is a non-increasing, nonnegative function, it is easy to

see that $g(x) x^k$ is slowly oscillating provided k is a positive integer. Indeed, for all $x < u < x(1 + \delta)$ we have

$$g(u) u^k \leq g(x) x^k (1 + \delta)^k.$$

Hence given $\varepsilon > 0$, any $\delta > 0$ such that $(1 + \delta)^k < 1 + \varepsilon$ would satisfy the condition.

Slow oscillation is generally a sufficient condition of Tauberian type theorems. For example, Theorem 17.2. of [9] states the following.

Theorem B. *Let f be defined on an interval (a, ∞), and suppose that $f(x) \sim Ax^\alpha$ as $x \to \infty$ with some real numbers α, A. If f is m times differentiable and*

$$\liminf \frac{f^{(m)}(y) - f^{(m)}(x)}{x^{\alpha - m}} \geq 0$$

as $x \to \infty$ and $1 < y/x \to 1$, then

$$f^{(j)}(x) \sim A\alpha(\alpha - 1) \ldots (\alpha - j + 1) x^{\alpha - j}$$

as $x \to \infty$, for $1 \leq j \leq m$.

Based on equation (32), we can apply this theorem with

$$f(x) = -\int_0^x g(u) u^k du, \quad m = j = 1, \quad \text{and} \quad \alpha = \gamma + k + 1.$$

Then we get that $g(x) x^{-\gamma}$ is convergent as $x \to \infty$.

From Theorem 2 it follows that the limit of $g(x) x^{-\gamma}$ is positive and finite, which is just our Theorem 3. $\square$

Remark 5. Since the Laplace transform method usually gives only local results in the discrete case, and we needed global results there, it is reasonable to use classic renewal techniques. On the other hand, those methods rely on convolution in the continuous case, which was not useful for our integral equation.

REFERENCES

[1] Á. Backhausz and T. F. Móri, A random model of publication activity. Manuscript. arXiv:1104.1026v4 [math.PR] http://arxiv.org/abs/1104.1026v4
[2] N. H. Bingham, *Regular variation*, Cambridge University Press, 1987.
[3] F. Chung and L. Lu, *Complex graphs and networks*, CBMS Regional Conference Series in Mathematics, 107, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006.
[4] C. Cooper, A. Frieze, A general model of web graphs, *Random Structures Algorithms*, **22** (2003), 311–335.
[5] N. G. de Bruijn and P. Erdős, On a recursion formula and on some Tauberian theorems, *J. Research Nat. Bur. Standards* **50** (1953), 161–164.

[6] W. Feller, *An introduction to probability theory and its applications. Vol. I*, 3rd ed., Wiley, New York, 1968.

[7] W. Feller, *An introduction to probability theory and its applications. Vol. II*, 2nd ed., Wiley, New York, 1971.

[8] G. H. Hardy, *Divergent Series*, Clarendon Press, Oxford, 1949.

[9] J. Korevaar, *Tauberian theory*, Grundlehren der Mathematischen Wissenschaften, 329, Springer, Berlin, 2004.

[10] P. Linz, *Analytical and numerical methods for Volterra equations.* SIAM Studies in Applied Mathematics, 7. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1985.

[11] I. M. Milne-Thompson, *The Calculus of Finite Differences*, MacMillan, London, 1933.

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